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Shear-flow instability at the interface between two viscous fluids

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We consider the linear stability of the cocurrent flow of two fluids of different viscosity in an infinite region (the viscous analogue of the classical Kelvin–Helmholtz problem). Attention is confined to the simplest case, Couette flow, and we solve the problem using both numerical and asymptotic techniques. We find that the flow is always unstable (in the absence of surface tension). The instability arises at the interface between the two fluids and occurs for short wavelengths, when viscosity rather than inertia is the dominant physical effect.

1. Introduction

When two fluids of different viscosities μ_1 and μ_2 are contained between two rigid plane-parallel boundaries, the interface being also plane and parallel to the boundaries and when the fluids are set in rectilinear motion, either by applied pressure gradient (Poiseuille flow), or by relative motion of the boundaries (Couette flow), a long-wavelength instability arises, which persists at arbitrarily small values of the Reynolds number (Yih 1967; Li 1969; Hickox 1971).‡ This instability is associated with the jump in viscosity across the interface; it is not clear, however, from previous studies whether the presence of the rigid boundaries plays an essential role in this instability (as it does in the classical problem of the stability of plane Poiseuille flow of a homogeneous fluid – see e.g. Drazin & Reid (1981, chap. 4)).

In this paper we examine the stability of the unbounded flow configuration, sketched in figure 1, in which the two fluids, identified by suffixes 1 and 2, occupy the half-spaces $y' > 0$ and $y' < 0$, the velocity field being

$$\mathbf{u} = \begin{cases} (a_1 y', 0, 0) & (y' > 0), \\ (a_2 y', 0, 0) & (y' < 0), \end{cases}$$

where, by virtue of continuity of tangential stress,

$$\mu_1 a_1 = \mu_2 a_2. \tag{1}$$

In this way we focus attention on any instability that is intrinsic to the interface, and that does not depend on the presence of rigid boundaries. We shall find that the interface may be unstable to *small*-wavelength perturbations (although of course surface tension exerts a stabilizing influence). The analysis is relevant in a local sense to any situation in which a shear flow acts in the neighbourhood of a viscosity jump.

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‡ The papers of Feldman (1957) and Tsahalis (1979) are also relevant; but these authors considered perturbations to the flow which do not perturb the interface, a severe restriction which limits the validity of their analyses.

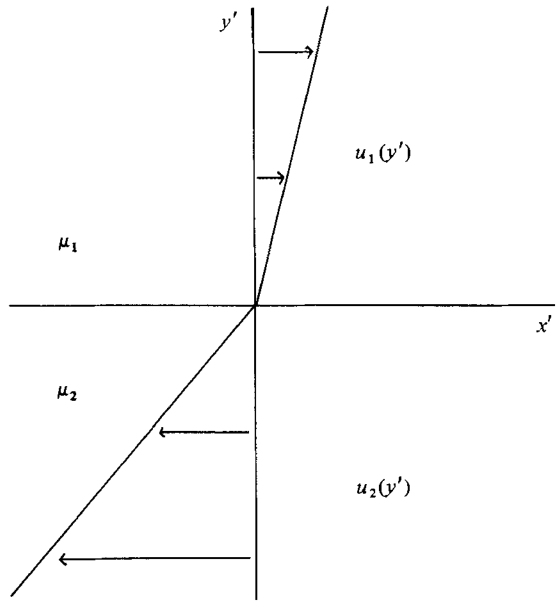

FIGURE 1. The cocurrent flow of two fluids, each in uniform shearing motion.

In §2 we define the problem, and in §3 we present a heuristic analysis which focuses attention on the nature of the short-wavelength instability. In §4, we obtain the exact dispersion relation for disturbances of arbitrary wavelength, for the particular case of equal densities ($\rho_1 = \rho_2$) and zero surface tension, and we solve this using numerical and asymptotic techniques. The results confirm the existence of a short-wavelength instability, and support the analysis of this instability presented in §3. In §5 we discuss the mechanism by which energy drives the instability.

2. Formulation of the problem

We consider the temporal evolution of a small disturbance applied to the flow described in §1. The appropriate equations and boundary conditions are discussed in detail at the beginning of Yih's (1967) paper.

First, following Yih, we non-dimensionalize with respect to the lower fluid by defining the following dimensionless variables:

$$(X, Y) = \left(\frac{\rho_2 a_2}{\mu_2}\right)^{\frac{1}{2}} (x', y'), \tag{2a}$$

$$t = a_2 t'. \tag{2b}$$

It is conventional in the literature to assume that the disturbance has X- and t-dependence of the form $\exp(i\alpha(X - Ct))$, and thus derive the Orr–Sommerfeld equations for the stream functions in each fluid (cf. Yih 1967, equations (21), (21a)). For our purposes it is more convenient to measure length on the scale of the wavelength, and this is the lengthscale we shall use in the rest of the paper. Accordingly we define rescaled coordinates (x, y) and a rescaled phase speed c by

$$(x, y) = \alpha(X, Y), \quad c = \alpha C. \tag{3}$$

We remark that the temporal growth rate of the disturbance is $\mathrm{Im}(c)$. One thus finds that with x- and t-dependence of the form $\exp(i(x-ct))$, the stream functions satisfy the Orr–Sommerfeld equations

$$\left(\frac{d^2}{dy^2}-1\right)^2\phi_1 = i\frac{m}{r}\alpha^{-2}(my-c)\left(\frac{d^2}{dy^2}-1\right)\phi_1, \quad (4a)$$

$$\left(\frac{d^2}{dy^2}-1\right)^2\phi_2 = i\alpha^{-2}(y-c)\left(\frac{d^2}{dy^2}-1\right)\phi_2 \quad (4b)$$

for the upper and lower fluids respectively. In (4a)

$$m = \frac{\mu_2}{\mu_1} = \frac{a_1}{a_2}, \quad r = \frac{\rho_2}{\rho_1}.$$

Note that $2\pi/\alpha$ is the ratio of the wavelength to the diffusion length $(\mu_2/\rho_2 a_2)^{\frac{1}{2}}$.

The boundary conditions at the interface of the fluids are that each component of velocity and stress is continuous. The form which these take in a problem such as ours is adequately discussed by Yih (1967) (see his equations (24), (26), (27), (30)). We require that on $y = 0$

$$\phi_1 - \phi_2 = 0, \quad (5a)$$

$$(\phi_1' - \phi_2') - \frac{1-m}{c}\phi = 0, \quad (5b)$$

$$(\phi_1'' + \phi_1) - m(\phi_2'' + \phi_2) = 0, \quad (5c)$$

$$(\phi_1''' - 3\phi_1') - m(\phi_2''' - 3\phi_2') = i\alpha^{-2}m\left[\left(1-\frac{1}{r}\right)(c\phi_2' + \phi_2) - \frac{\alpha^3 S}{1-m}(\phi_1' - \phi_2')\right]. \quad (5d)$$

Our problem is thus, for any given real α, to find non-trivial solutions (or eigenfunctions) ϕ_j of the Orr–Sommerfeld equations (4a, b) subject to the interface conditions (5a–d) and to the requirement that ϕ_j tend to zero as $y \to \infty$ ($j = 1$) or $-\infty$ ($j = 2$). It suffices to consider only the case $m < 1$, since if $\phi(y, \alpha, m, r, S)$ is an eigenfunction corresponding to the eigenvalue $c(\alpha, m, r, S)$, it may readily be seen from (4) and (5) that

$$\phi^*\left(-y, \frac{\alpha r^{\frac{1}{2}}}{m}, \frac{1}{m}, \frac{1}{r}, \frac{\alpha r^{\frac{1}{2}} S}{m}\right)$$

is also an eigenfunction, with corresponding eigenvalue

$$-mc^*\left(\frac{\alpha r^{\frac{1}{2}}}{m}, \frac{1}{m}, \frac{1}{r}, \frac{r^{\frac{1}{2}} S}{m}\right),$$

where $*$ denotes complex conjugate.

3. A regular perturbation analysis of the unstable mode

3.1. *The perturbation scheme*

The form of the Orr–Sommerfeld equations (4a, b) suggests that as $\alpha \to \infty$, that is in the short-wave limit, there might exist solutions of the form

$$\phi_1 = \left(a_0(y) + \frac{a_1(y)}{\alpha^2} + \frac{a_2(y)}{\alpha^4} + \ldots\right)e^{-y}, \quad (6a)$$

$$\phi_2 = \left(b_0(y) + \frac{b_1(y)}{\alpha^2} + \frac{b_2(y)}{\alpha^4} + \ldots\right)e^{y}, \quad (6b)$$

$$c = c_0 + \frac{c_1}{\alpha^2} + \frac{c_2}{\alpha^4} + \ldots. \quad (6c)$$

These solutions cannot be expected to be uniformly valid in y: indeed they may be expected to fail when $|y|$ becomes comparable to, or larger than, α^2. (The factors e^{-y} and e^y are not strictly essential to our argument, but it is convenient to introduce them now: it turns out that each $a_n(y)$, $b_n(y)$, $n = 0, 1, 2, 3, \ldots$, is a polynomial of degree $2n+1$.) When the ansatz (6) is substituted into the differential equations and the boundary conditions, the result in each case is a descending power series in α^{-2}, each coefficient of which must vanish. Thus we arrive at an iterative scheme whereby $a_n(y)$, $b_n(y)$, c_n are found at each state in terms of the previously calculated iterates.

Substitution of (6) in the Orr–Sommerfeld equations (4) yields the fourth-order differential equations

$$a_n^{\mathrm{iv}}(y) - 4a_n'''(y) + 4a_n''(y) = i\frac{m^2}{r}y(a_{n-1}''(y) - 2a_{n-1}'(y)) - i\frac{m}{r}\sum_{s=0}^{n-1} c_s(a_{n-1-s}''(y) - 2a_{n-1-s}'(y)), \quad (7a)$$

$$b_n^{\mathrm{iv}}(y) + 4b_n'''(y) + 4b_n''(y) = iy(b_{n-1}'' + 2b_{n-1}'(y)) - i\sum_{s=0}^{n-1} c_s(b_{n-1-s}''(y) + 2b_{n-1-s}'(y)), \quad (7b)$$

where $n = 0, 1, 2, \ldots$. In this equation and in all subsequent work, functions with negative indices ($a_{-1}(y)$ etc.) are to be understood to be identically zero.

Now consider the boundary conditions on $y = 0$. Continuity of normal velocity ($5a$) implies

$$a_n(0) = b_n(0) \quad (n = 0, 1, 2, \ldots). \quad (8a)$$

Continuity of tangential velocity ($5b$) implies

$$\sum_{s=0}^{n} c_s(a_{n-s}'(0) - b_{n-s}'(0) - 2a_{n-s}(0)) - (1-m)\,a_n(0) = 0 \quad (n = 0, 1, 2, \ldots). \quad (8b)$$

From this equation, c_n may be determined if all the previous iterates are known. (Continuity of tangential stress ($5c$) yields

$$(a_n''(0) - 2a_n'(0) + 2a_n(0)) - m(b_n''(0) + 2b_n'(0) + 2b_n(0)) = 0 \quad (n = 0, 1, 2, \ldots), \quad (8c)$$

and continuity of normal stress ($5d$) yields

$$(a_n'''(0) - 3a_n''(0) + 2a_n(0)) - m(b_n'''(0) + 3b_n''(0) - 2b_n(0)) = im\left(1 - \frac{1}{r}\right)\left(\sum_{s=0}^{n-1} c_s(b_{n-1-s}'(0) + b_{n-1-s}(0)) + b_{n-1}(0)\right) - i\frac{m\alpha^3 S}{1-m}(a_{n-1}'(0) - b_{n-1}'(0) - 2a_{n-1}(0)) \quad (n = 0, 1, 2, \ldots). \quad (8d)$$

There is clearly a difficulty concerning the order of magnitude of the surface-tension term. The condition ($8d$) is derived under the assumption that $\alpha^3 S = O(1)$ for large α. As we shall see, the iterative scheme works satisfactorily with the boundary condition ($8d$). If, however, one were to assume that $\alpha^3 S$ were larger than this, say $\alpha^3 S = O(\alpha^2)$, then the surface-tension term in ($8d$) would involve a_n and b_n and one would quickly find that the method would fail, the failure manifesting itself in the appearance of a set of inconsistent linear equations. We shall return to this point later.

Finally we consider the conditions at $y = \pm\infty$. Since we are requiring the stream function to tend to zero there, we shall demand

$$a_n(y) = o(e^y) \quad (y \to \infty), \quad (9a)$$

$$b_n(y) = o(e^{-y}) \quad (y \to -\infty). \quad (9b)$$

We remark that the perturbation method described above has some similarities with that given by Yih (1963) on a related problem (see his §§IV, V).

3.2. *Equal density and zero surface tension*

In the case to be considered in this section we take $r = 1$ and $S = 0$. We study this somewhat artificial situation in some detail because the problem is then at its simplest. The analysis may be readily extended to more realistic situations, and we describe the result of doing so in §3.3.

Setting $n = 0$ in the differential equations (7) and taking account of ($9a, b$), we find that the zeroth-order solutions are of the form

$$a_0(y) = \alpha_{00} + \alpha_{01} y,$$

$$b_0(y) = \beta_{00} + \beta_{01} y$$

for some constant coefficients $\alpha_{00}, \alpha_{01}, \beta_{00}, \beta_{01}$. The boundary conditions ($8a, d$) – those associated with the continuity of normal velocity and normal stress – together imply

$$\alpha_{00} = \beta_{00} = 0.$$

The boundary condition ($8c$) – the condition associated with the continuity of tangential stress – implies that

$$\alpha_{01} + m\beta_{01} = 0.$$

Let $\beta_{01} = \lambda_0$, an arbitrary constant. Then one finds

$$a_0(y) = -m\lambda_0 y, \quad b_0(y) = \lambda_0 y. \qquad (10a, b)$$

Finally ($8b$) – the tangential velocity condition – implies

$$c_0 = 0. \qquad (10c)$$

This result implies nothing about the stability of the mode.

Now we proceed to the first-order analysis. The differential equations are

$$a_1^{\text{iv}}(y) - 4a_1'''(y) + 4a_1''(y) = 2im^3\lambda_0 y \quad (y > 0),$$

$$b_1^{\text{iv}}(y) + 4b_1'''(y) + 4b_1''(y) = 2i\lambda_0 y \quad (y < 0).$$

Thus we find

$$a_1(y) = \alpha_{10} + \alpha_{11} y + \tfrac{1}{4}im^3\lambda_0 y^2 + \tfrac{1}{12}im^3\lambda_0 y^3,$$

$$b_1(y) = \beta_{10} + \beta_{11} y - \tfrac{1}{4}i\lambda_0 y^2 + \tfrac{1}{12}i\lambda_0 y^3$$

for some coefficients $\alpha_{10}, \alpha_{11}, \beta_{10}, \beta_{11}$. The boundary conditions ($8a, d$) imply

$$\alpha_{10} = \beta_{10} = -\tfrac{1}{2}im(1-m)\lambda_0.$$

The tangential-stress boundary condition ($8c$) implies that the remaining two coefficients satisfy

$$\alpha_{11} + m\beta_{11} = -\tfrac{1}{4}im(m^2 - 4m + 1)\lambda_0. \qquad (11)$$

This equation clearly has no unique solution; furthermore one can readily see that at each stage of the iteration the equations associated with the continuity of tangential stress will always have this property. We shall now turn aside from our discussion of the iterative scheme to discuss this difficulty.

We can, in analogy with λ_0, introduce the arbitrary constant λ_1, so that the solution of (11) is

$$\beta_{11} = \lambda_1, \quad \alpha_{11} = -m\lambda_1 - \tfrac{1}{4}im(m^2 - 4m + 1)\lambda_0.$$

Likewise, at the next stage of the iteration, one can introduce the arbitrary constant λ_2, so that $\beta_{21} = \lambda_2$. Such constants $\lambda_1, \lambda_2, \lambda_3, \ldots$ can be introduced at each stage of the iteration. It may be deduced from the linearity of the problem that the eigenfunction will be expressed as a linear combination of the arbitrary constants $\lambda_1, \lambda_2, \lambda_3, \ldots$, and further that each coefficient of $\lambda_1, \lambda_2, \lambda_3, \ldots$ differs from its predecessor only by the factor α^{-2}; the eigenvalue is independent of the choice of $\lambda_1, \lambda_2, \lambda_3, \ldots$. The arbitrary constants may be regarded as a consequence of the fact that any eigenfunction may be multiplied by an arbitrary function of α.

Returning now to (11), it is clear from the preceding discussion that the particular choice of solution has no great significance. We choose

$$\alpha_{11} = -\tfrac{1}{4}im(m^2-2m)\lambda_0, \quad \beta_{11} = -\tfrac{1}{4}i(-2m+1)\lambda_0, \tag{12}$$

so that we find

$$a_1(y) = -\tfrac{1}{2}im(1-m)\lambda_0 - \tfrac{1}{4}im(m^2-2m)\lambda_0 y + \tfrac{1}{4}im^3\lambda_0 y + \tfrac{1}{12}im^3\lambda_0 y^3, \tag{13a}$$

$$b_1(y) = -\tfrac{1}{2}im(1-m)\lambda_0 - \tfrac{1}{4}i(-2m+1)\lambda_0 - \tfrac{1}{4}i\lambda_0 y^2 + \tfrac{1}{12}i\lambda_0 y^3. \tag{13b}$$

Finally we use the condition (8*b*) to show

$$c_1 = i\frac{m(1-m)^2}{2(1+m)}. \tag{13c}$$

This last result implies that, for sufficiently large α, the mode is unstable for all values of m, other than $m = 1$ (corresponding to unbounded Couette flow of a single fluid), or $m = 0$.

One may then proceed in a similar fashion to calculate the higher-order approximations to the eigenfunctions and thence to the eigenvalues. One finds that the second-order correction to the eigenvalue is

$$c_2 = \frac{-m(1-m)}{16(1+m)^2}[5m^4+12m^3-20m^2+12m+5]. \tag{14}$$

The expression in square brackets is positive for all values of the ratio m. (For example it may be written as $5(1-m)^4+32m(m-1)^2+14m^2$.) Thus c_2 is real, and is positive or negative according as $m \gtrless 1$.

From (10*c*), (13*c*), (14), we deduce that

$$c = i\frac{m(1-m)^2}{2(1+m)}\alpha^{-2} - \frac{m(1-m)}{16(1+m)^2}(5m^4+12m^3-20m^2+12m+5)\alpha^{-4}+O(\alpha^{-6}), \tag{15}$$

where, of course, the error term is merely a formal statement about the nature of the ansatz (6), and not in any sense a rigorous bound. The result we have just derived agrees satisfactorily with the separate numerical calculations of §4.

3.3. *Unequal density and non-zero surface tension, and further extensions*

The effect of unequal density ($r \neq 1$) changes the nature of the problem we have just discussed in two ways. First the right-hand side of the differential equation (7*a*) now has $1/r$ factors (instead of unity), and secondly the first term on the right-hand side of the normal-stress condition (8*d*) no longer vanishes. The effect of non-zero surface tension manifests itself only in the normal-stress condition: now the second term in (8*d*) no longer vanishes.

It is clear that the iterative scheme will proceed successfully as before because the terms on the right-hand sides of (7*a*) and (8*b*) have each been obtained at previous stages of the iteration. At the zeroth-order level, the problem is unchanged from that considered in §3.2, and so we again recover results (10*a*–*c*). At the first-order level, however, the differential equation ($7a$) changes to the extent of a factor $1/r$ on the right-hand side. Working through the analysis, one finds

$$c_1 = i\frac{m(1-m)(1-m^2/r)}{2(1+m)^2} - i\frac{m}{2(1+m)}\alpha^3 S. \qquad (16)$$

Should one wish to do so, one may proceed to higher-order contributions, but the result (16) is sufficient to discuss the stability of the mode. We observe first that, as one expects, the effect of surface tension is always stabilizing for all values of m. Now consider the effect of density variation. To fix our ideas, suppose $m < 1$, i.e. the lower fluid is the less viscous. Then if the lower fluid is the more dense ($r > 1$) the instability is reinforced by the density difference; if the lower fluid is the less dense then the density difference is a stabilizing influence and, in the absence of surface tension, will actually produce stability when $r < m^2$.

The asymptotic analysis is capable of extension in other directions. Hooper (1981) considers the effect of gravity (an external body force $\rho_j g$ in the direction of negative y'). She finds that $c_0 = 0$ again, but now the term

$$-i\frac{m}{2(1+m)}\left(1-\frac{1}{r}\right)g\left(\frac{\rho_2}{a_2^3\mu_2}\right)^{\frac{1}{2}}$$

has to be added to the right-hand side of (16).

A further significant extension of the results is possible when it is appreciated that in the short-wave limit, for the particular mode discussed in this part of the paper, the disturbance is localized in the neighbourhood of the interface, and that spatial variations occur on the scale of the wavelength. Now for sufficiently short waves, any parallel shear flow with velocity components $(U_j(y'), 0)$, $U_j'(0) \neq 0$, is locally Couette near the interface. Thus we should expect very similar results to hold for any such flow.

One can place the preceding argument on a more formal basis as follows. Let the flow field be $(U_j(y'), 0)$, $j = 1$ or 2, and let $U_1(0) = U_2(0)$. We again require continuity of tangential stress (1), where now

$$a_1 = U_1'(0), \quad a_2 = U_2'(0).$$

Instead of (4), one now finds

$$\left(\frac{d^2}{dy^2}-1\right)^2\phi_1 = i\frac{m}{r}\alpha^{-2}(u_1(y)-c)\left(\frac{d^2}{dy^2}-1\right)\phi_1 - i\frac{m}{r}\alpha^{-2}u_1''(y)\,\phi_1,$$

$$\left(\frac{d^2}{dy^2}-1\right)^2\phi_2 = i\alpha^{-2}(u_2(y)-c)\left(\frac{d^2}{dy^2}-1\right)\phi_2 - i\alpha^{-2}u_2''(y)\,\phi_2,$$

where length and time are rescaled as before. In particular

$$u_j(y) = \alpha\left(\frac{\rho_2}{a_2\mu_2}\right)^{\frac{1}{2}}U_j(y') \quad \text{when} \quad y = y' \quad (j = 1, 2).$$

The interface conditions ($5a$–d) remain unchanged.

Now $u_j''(y)$ measures the second derivative of the scaled velocity with respect to y, a variable which measures length on the scale of the very small wavelength of the disturbance. So unless the velocity profile is extraordinarily curved, it is reasonable to assume $u_j''(y) = O(\alpha^{-1})$. Thus to the zeroth and first orders, the problem is the same as before, and so we again recover ($13c$). We remark that if one were to proceed to higher orders it would, in general, no longer be appropriate to seek the descending power series ($6a$–c) in powers of α^2, but instead a descending power series in α.

4. The exact solution

4.1. *The secular equation*

The problem posed in §2 may alternatively be tackled by finding the exact form of the eigenfunctions. We restrict our attention to the case of equal density, though we allow the surface tension to be non-zero.

Denote the vorticity of the disturbance in each fluid by $\omega_j(y)\, e^{i(x-ct)}$, so that

$$\omega_j(y) = -\left(\frac{d^2}{dy^2}-1\right)\phi_j \quad (j = 1,2). \tag{17}$$

Then the Orr–Sommerfeld equations may be written

$$\frac{d^2\omega_1}{dy^2} - im\alpha^{-2}(my-c-im^{-1}\alpha^2)\,\omega_1 = 0 \quad (y>0), \tag{18a}$$

$$\frac{d^2\omega_2}{dy^2} - i\alpha^{-2}(y-c-i\alpha^2)\,\omega_2 = 0 \quad (y<0). \tag{18b}$$

Equations (18) have the respective solutions

$$\mathrm{Ai}\,(\alpha^{-\frac{2}{3}}m^{\frac{2}{3}}(y-m^{-1}c-im^{-2}\alpha^2)\,e^{i\theta_1}), \quad \mathrm{Ai}\,(\alpha^{-\frac{2}{3}}(y-c-i\alpha^2)\,e^{i\theta_2})$$

in $y>0$ and $y<0$, where $\theta_j = \frac{1}{6}\pi,\ \frac{5}{6}\pi$, or $-\frac{1}{2}\pi$. The vorticity of the perturbed flow must tend to zero as $y\rightarrow \pm\infty$; this condition can only be satisfied if $\theta_1 = \frac{1}{6}\pi$, $\theta_2 = \frac{5}{6}\pi$. In order that our notation be a simple as possible, we write

$$A_1(y) = \mathrm{Ai}\,(\alpha^{-\frac{2}{3}}m^{\frac{2}{3}}(y-m^{-1}c-im^{-2}\alpha^2)e^{\frac{1}{6}i\pi}), \tag{19a}$$

$$A_2(y) = \mathrm{Ai}\,(\alpha^{-\frac{2}{3}}(y-c-i\alpha^2)\,e^{\frac{5}{6}i\pi}). \tag{19b}$$

Thus

$$\omega_j = b_j\,A_j(y) \quad (j = 1,2)$$

for some constants b_1, b_2. With these expressions for the vorticity, the stream functions are found from (17) to be

$$\Phi_1 = a_1\,e^{-y} + b_1\left[e^{-y}\int_0^y e^z A_1(z)\,dz + e^y\int_y^\infty e^{-z}A_1(z)\,dz\right], \tag{20a}$$

$$\Phi_2 = a_2\,e^{y} + b_2\left[e^{y}\int_0^y e^{-z} A_2(z)\,dz + e^{-y}\int_y^{-\infty} e^{z}A_2(z)\,dz\right] \tag{20b}$$

for some constants a_1, a_2.

When the results (20) are substituted into each of the boundary conditions (5*a–d*), one finds four homogeneous linear equations for the four unknowns a_1, a_2, b_1, b_2; namely

$$\begin{pmatrix} 1 & -1 & J_1 & J_2 \\ -1+\dfrac{m}{c} & -1-\dfrac{1}{c} & \left(1+\dfrac{m}{c}\right)J_1 & \left(-1+\dfrac{1}{c}\right)J_2 \\ 2 & -2m & 2J_1-2A_1 & -m(-2J_2+2A_2) \\ 2-\dfrac{i\alpha mS}{1-m} & 2m-\dfrac{i\alpha mS}{1-m} & \begin{matrix}-2J_1-2A_1' \\ +i\dfrac{\alpha mS}{1-m}J_1\end{matrix} & \begin{matrix} m(2J_2-2A_2') \\ -i\dfrac{\alpha mS}{1-m}J_2\end{matrix} \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ b_1 \\ b_2 \end{pmatrix} = 0, \tag{21}$$

where

$$J_1 = \int_0^\infty e^{-z}A_1(z)\,dz, \quad J_2 = \int_{-\infty}^0 e^{z}A_2(z)\,dz,$$

$$A_j = A_j(0), \quad A_j' = \left.\frac{dA_j}{dy}(y)\right|_{y=0} \quad (j = 1,2).$$

In order that (21) have a non-trivial solution the determinant of the matrix must vanish: this yields the secular equation for the eigenvalue c. After a considerable amount of manipulation, the secular equation can be shown to be

$$F(m,\alpha,c,S) = 0, \tag{22}$$

where

$$F = 4(1-m)^2 J_1 J_2 + 2(1-m)\, m\left(A_2 + \frac{A_2}{c} + A_2'\right) J_1 + 2(1-m)\left(-A_1 + \frac{mA_1}{c} + A_1'\right) J_2 + m\left(2 + \frac{1-m}{c}\right)(A_1' A_2 - A_2' A_1) - \frac{i\alpha m S}{c}(J_2 A_1 + mJ_1 A_2). \tag{23}$$

To within a multiplicative constant, the coefficients may be shown to be

$$\left.\begin{aligned} a_1 &= (1-m) J_1 J_2 - A_1 J_2 + \frac{m(1-m)}{2c} A_2 J_1, \\ a_2 &= -(1-m) J_1 J_2 - mA_2 J_1 + \frac{1-m}{2c} A_1 J_2, \\ b_1 &= -(1-m) J_2 - mA_2 - \frac{m(1-m)}{2c} A_2, \\ b_2 &= -(1-m) J_1 + A_1 + \frac{1-m}{2c} A_1. \end{aligned}\right\} \tag{24}$$

The calculation whose results, (20) and (22), we have just reported may also be carried out when the densities of the fluids are unequal.

4.2. *Numerical solution of the secular equation*

We compute the Airy functions of complex argument that appear in (23) using an algorithm developed by Schulten, Anderson & Gordon (1979). We find the roots of the secular equation by searching for the minima of $|F|^2$ with respect to c for any given m, α and S.

More specifically our strategy is the following:

(i) fix the values of m, S and find the roots of the secular equation for $\alpha \ll 1$;

(ii) use the values of the roots found at the previous stage as starting values for a program in which, by increasing α by small steps, the manner in which each root varies with α is found for the fixed value of m;

(iii) when α becomes large, check that the numerical values of the roots thus found agree with the asymptotic results (which we shall derive in §4.3).

Initially we take surface tension S to be zero.

The asymptotic behaviour as $\alpha \rightarrow 0$ of (22) is found to be dominated by the term

$$\left(2 + \frac{1-m}{c}\right)(A_1' A_2 - A_2' A_1).$$

Thus as $\alpha \rightarrow 0$, the eigenvalues c are either such that

$$c \sim -\tfrac{1}{2}(1-m), \tag{25}$$

or they satisfy

$$A_1' A_2 - A_2' A_1 \sim 0.$$

This last equation has an infinite number of solutions: the asymptotic behaviour of the Airy functions for large values of their arguments yields the approximate solutions

$$c \sim (\tfrac{3}{2}\pi n - \tfrac{3}{8}\pi - \tfrac{3}{2}i \text{ artanh}^{-1} m^{\frac{1}{2}})^{\frac{2}{3}} e^{-\frac{1}{6}i\pi} m^{\frac{1}{2}} \alpha^{\frac{2}{3}}, \tag{26a}$$

$$c \sim (\tfrac{3}{2}\pi n - \tfrac{9}{8}\pi + \tfrac{3}{2}i \text{ artanh}^{-1} m^{\frac{1}{2}})^{\frac{2}{3}} e^{-\frac{5}{6}i\pi} \alpha^{\frac{2}{3}} \tag{26b}$$

n	$\kappa_{+n}(0)$	$\kappa_{-n}(0)$
1	$1{\cdot}375-1{\cdot}318i$	$-0{\cdot}604-1{\cdot}294i$
2	$2{\cdot}635-1{\cdot}920i$	$-2{\cdot}585-2{\cdot}056i$
3	$3{\cdot}645-2{\cdot}448i$	$-3{\cdot}980-2{\cdot}762i$
4	$4{\cdot}530-2{\cdot}925i$	$-5{\cdot}165-3{\cdot}392i$
5	$5{\cdot}336-3{\cdot}367i$	$-6{\cdot}225-3{\cdot}969i$

TABLE 1. The values of $\kappa_{\pm n}(0)$ for various values

for large positive integer n. Solutions of the type (26a) are closely associated with the zeros of the Airy functions A_1, A_1': those of the type (26b) with A_2, A_2'. (These assertions are more convincing if one considers the dimensional form of the eigenfunctions.) Each of (26a, b) may be used as starting values in the numerical scheme used to determine zeros of $F(m, \alpha, c, 0)$. The form of (19a, b) strongly suggests making the change of scale

$$\kappa = \alpha^{-\frac{2}{3}}c$$

for the eigenvalue c. When this is done, one does indeed find values of κ close to the approximate values in (26a, b). The predictions of (26) are quite accurate even for small n. We denote the roots corresponding to (26a) by $\kappa_{+n}(\alpha)$ and those corresponding to (26b) by $\kappa_{-n}(\alpha)$ $(n = 1, 2, 3, \ldots)$. Values of $\kappa_{\pm n}(0)$ for $m = 0{\cdot}5$ and $n = 1, 2, 3$ are shown in table 1. Note that the ordering of the $\kappa_{+n}(0)$ and $\kappa_{-n}(0)$ are such that, for each group of modes, the imaginary parts of the eigenvalues decrease with increasing n.

One also finds a root of the secular equation corresponding to the approximate solution (25): we denote it by $\kappa_0(\alpha)$. The result (25) yields information about only the real part of κ_0. An estimate of the imaginary part can be obtained by direct asymptotic analysis, for $\alpha \to 0$, of the secular equation, or by application of the WKB method. Both methods yield the result

$$c \sim -\tfrac{1}{2}(1-m) - i\frac{2(1+m)}{m}\alpha^2. \tag{27}$$

Thus one expects that, as $\alpha \to 0$, the real part of κ_0 will be large (and negative for $m < 1$) while the imaginary part will tend to zero through negative values. The numerical results quantitatively confirm these expectations. We remark for future reference that (27) is not uniformly valid near $m = 0$.

Having found the values of $\kappa_{\pm n}$ for $n = 0, 1, 2, \ldots$ when $\alpha = 0$, we can proceed to calculate $\kappa_{\pm n}(\alpha)$ as α increases in small steps. As $\alpha \to \infty$ (which in practice means α greater than about 2 or 3) it is found that each mode settles into one of five types of behaviour. These are discussed in §4.3; here we list them.

(1) $c \sim \frac{1}{8}im(1-m)^2(1+m)^{-1}\alpha^{-2}$, the single unstable mode discussed in §3.

(2) $c \sim -i\alpha^2/m + e^{\frac{5}{6}\pi i}a_q m^{\frac{1}{3}}\alpha^{\frac{2}{3}}$, where $q = 1, 2, 3, \ldots$. Here a_q is the qth zero of the Airy function, $\mathrm{Ai}(a_q) = 0$; there are an infinite number of such zeros, all of which are real and negative: correspondingly there are an infinite number of such modes.

(3) $c \sim -i\alpha^2 + e^{\frac{1}{6}\pi i}a_q\alpha^{\frac{2}{3}}$, where $q = 1, 2, 3, \ldots$, so that there are an infinite number of such modes.

(4) A single mode which is such that $c \sim \lambda_1\alpha^2$ for some complex constant λ_1; c is not asymptotic to either $-i\alpha^2/m$ or $-i\alpha^2$.

(5) Another single mode which is such that $c \sim \lambda_2\alpha^2$ for some complex constant

	$0 < m < 0{\cdot}181$	$0{\cdot}181 < m < 0{\cdot}23$	$0{\cdot}23 < m < 1$
Group 1	κ_{-1}	κ_{-2}	κ_0
Group 2	κ_{+n} $(n \geqslant 2)$	κ_{+n} $(n \geqslant 2)$	κ_{+n} $(n \geqslant 2)$
Group 3	κ_0, κ_{-n} $(n \geqslant 3)$	κ_0, κ_{-n} $(n \geqslant 3)$	κ_{-n} $(n \geqslant 2)$
Group 4	κ_{+1}	κ_{+1}	κ_{+1}
Group 5	κ_{-2}	κ_{-1}	κ_{-1}

TABLE 2. The correspondence between the behaviour of the modes as $\alpha \to \infty$ and as $\alpha \to 0$

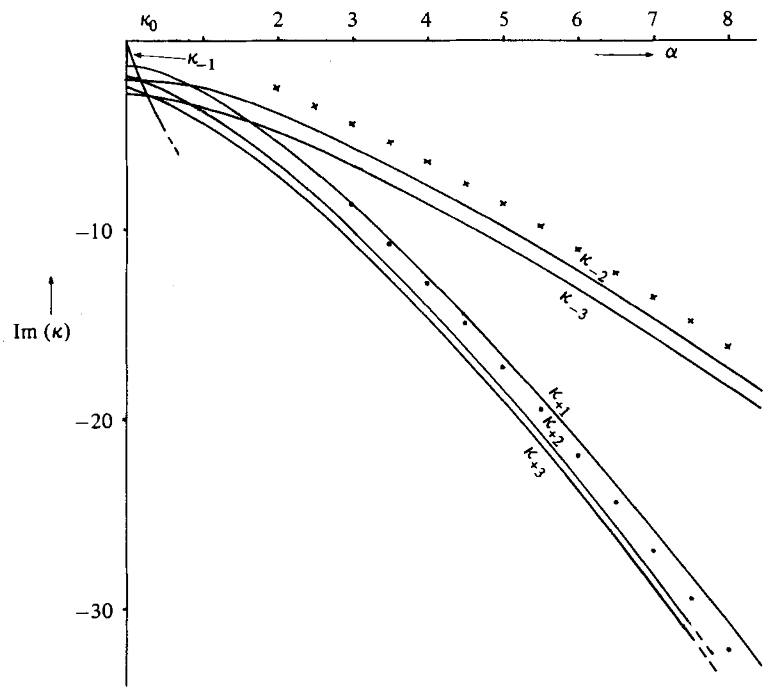

FIGURE 2. The variation of $\text{Im}(\kappa)$ with α of the first seven modes when $m = 0{\cdot}5$ (the κ_0 mode is barely visible on this scale). The points marked with × and ● denote values of $-\alpha^{\frac{4}{3}}$ and $-\alpha^{\frac{4}{3}}/m$ respectively.

λ_2; c is not asymptotic to either $-i\alpha^2/m$ or $-i\alpha^2$, and one finds $|\lambda_2|$ is much greater than $|\lambda_1|$.

We shall refer to these as group 1 modes, group 2 modes, and so on.

The numerical results we have obtained imply that, when $m = 0{\cdot}181$ or $0{\cdot}23$, mode-crossing occurs. The correspondence between the two asymptotic regimes is summarized in table 2.

The numerical results we have found are well illustrated in figure 2, where $\text{Im}\,(\kappa_{\pm n})$ for $n = 0, 1, 2$ is plotted against α for the particular value $m = 0{\cdot}5$. The results are typical of those obtained for $m > 0{\cdot}23$.

We were unable to compute values of $\kappa_{-1}(\alpha)$ for values of α greater than $0{\cdot}5$ for any values of the viscosity ratio $m > 0{\cdot}181$. As is evident from figure 2, $\text{Im}\,\kappa_{-1}$ dips sharply. The difficulty is that, even for relatively small values of α, the arguments of the Airy functions in the secular equation are large, and the computations become slow and inefficient. The same difficulty arises from $m < 0{\cdot}181$ in the computation of $\kappa_{-2}(\alpha)$.

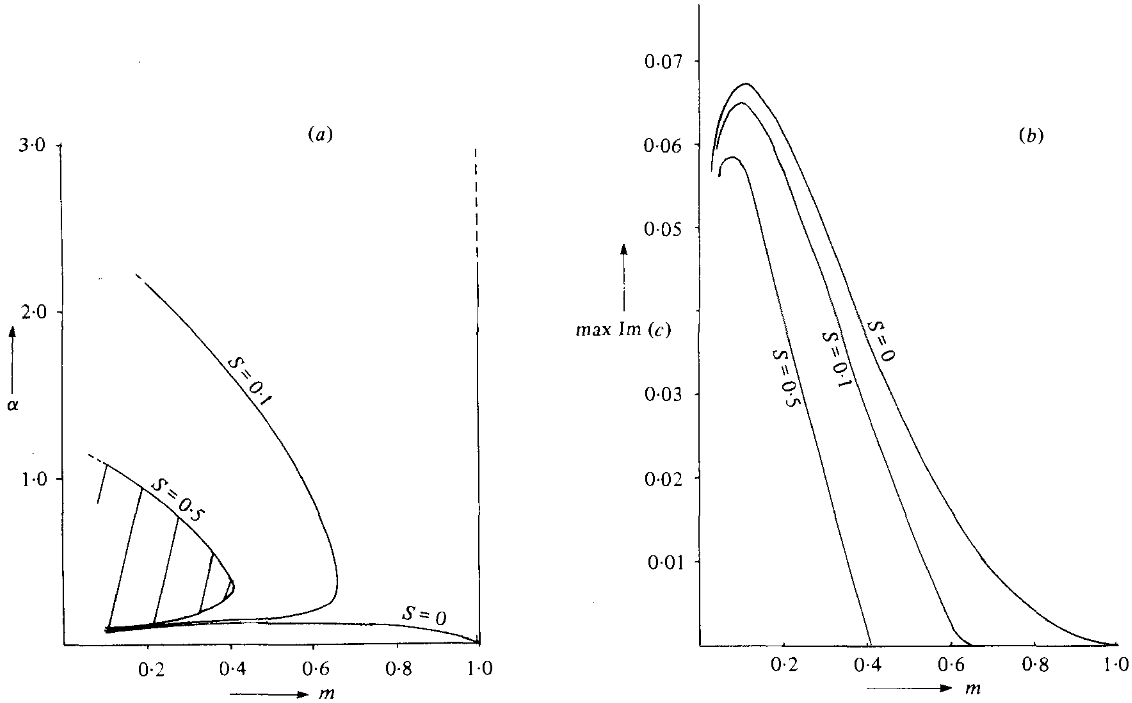

FIGURE 3. (a) The marginal stability curves for different values of the surface tension S. The shaded area denotes the region of instability when $S = 0{\cdot}5$. The regions of instability for the other values of S are analogous and may be readily envisaged. Note the singular nature of the marginal stability curve when $S = 0$. (b) The plotted curves represent, for specific values of the surface tension S, the maximum growth rate of the instability as a function of m.

The mode that is of physical interest is the unstable mode – the group 1 mode. The properties of this mode are conveniently summarized in figures 3 (a, b) for various values of the surface tension S. There are difficulties associated with each of the limits $m \rightarrow 1$ and $m \rightarrow 0$, which we now discuss.

As $m \rightarrow 1$ when $S = 0$, it is clear from the marginal stability curve 3 (a) that instability is eventually present for all wavenumbers other than $\alpha = 0$. When $m = 1$, however, the flow is unbounded Couette flow of a single fluid, a flow that is always stable (Marcus & Press 1977). Reference to figure 3 (b) resolves the paradox: the maximum growth rate of the instability tends to zero as $m \rightarrow 1$.

We were unable to plot the graphs of figures 3 (a, b) for small values of m because the numerical calculations were then inefficient. Some progress can be made by allowing $m \rightarrow 0$ and assuming $c = O(m)$ in the secular equation (22). When account is taken of surface tension this yields, to leading order,

$$c \sim m\left(1 - \frac{A_2'}{2J_2} - \frac{i\alpha S}{2}\right) \quad \text{as} \quad m \rightarrow 0 \tag{28}$$

with $c = 0$ in the definitions of the functions on the right-hand side. The requirement that the right-hand side of (28) be real yields a value of α for any given non-zero S, namely the value of α at which the upper branch of the marginal stability curve intersects the $m = 0$ axis. For the values of S shown in figure 3 (a) one finds $\alpha = 2{\cdot}121$ ($S = 0{\cdot}1$) and $\alpha = 1{\cdot}181$ ($S = 0{\cdot}5$). More detailed information about the behaviour of the marginal stability curve near $m = 0$ can obviously be obtained by considering higher-order terms in (28).

The formula (28) yields no information about the lower branches of the marginal stability curves as $m \to 0$. The reason for the failure is that, on the lower branches, α and m are small simultaneously, and (28) is not uniformly valid for small α. We infer from (27) and (28), however, that the lower branches of the marginal stability curves cannot pass through the α-axis or m-axis except at the origin. It may be that a more complicated analysis in the simultaneous limits $\alpha \to 0$, $m \to 0$ will yield more satisfactory results.

4.3. *Asymptotic behaviour as* $\alpha \to \infty$

We again consider only the case $S = 0$.

To find the asymptotic behaviour as $\alpha \to \infty$ of the eigenvalues c (the solutions of the secular equation $F(m, \alpha, c, 0) = 0$) one first finds the asymptotic expansion of $F(m, \alpha, c, 0)$ as $\alpha \to \infty$, $F \sim G(m, \alpha, c)$ say, and then solves $G(m, \alpha, c) = 0$. A difficulty immediately manifests itself when one attempts to find G: it is that the asymptotic behaviour of $F(m, \alpha, c, 0)$ as $\alpha \to \infty$ depends on the value of c – which we are trying to find. Thus consider the term A_2 which occurs in (22), i.e.

$$\mathrm{Ai}\,(\alpha^{-\frac{2}{3}}(c+i\alpha^2)^{-\frac{1}{6}\pi i}). \tag{29}$$

The Airy function has the asymptotic expansion

$$\mathrm{Ai}\,(z) = \tfrac{1}{2}\pi^{-\frac{1}{2}}z^{-\frac{1}{4}}\exp\,(-\tfrac{2}{3}z^{\frac{3}{2}})\,(1+O(z^{-\frac{3}{2}})) \tag{30}$$

as $|z| \to \infty$, where $|\arg z| < \pi$ (Abramowitz & Stegun 1965, 10.4.59); there is a different asymptotic expansion valid as $|z| \to \infty$ in domains which include the negative real axis (Abramowitz and Stegun (1965), 10.4.60); if $|z|$ is not large, no simplification is possible at all. Define

$$\epsilon_2 = \alpha^{-\frac{2}{3}}(c+i\alpha^2)\,e^{-\frac{1}{6}\pi i}. \tag{31a}$$

Then, depending on the value taken by ϵ_2 (and so, by implication, c), each of these three regimes occurs for the Airy function given in (29).

Similar remarks apply to A_2' and to A_1, A_1' when one defines

$$\epsilon_1 = \alpha^{-\frac{2}{3}}m^{-\frac{1}{3}}\left(c+i\frac{\alpha^2}{m}\right)e^{-\frac{5}{6}\pi i}. \tag{31b}$$

The same difficulty manifests itself for the integrals that occur in (23). Thus consider J_2: a simple change of variable yields

$$J_2 = \alpha^{\frac{2}{3}}\int_0^\infty \exp\,[-\alpha^{\frac{2}{3}}w]\,\mathrm{Ai}(e^{-\frac{1}{6}\pi i}w+\epsilon_2)\,dw. \tag{32}$$

Now if $\epsilon_2 = O(1)$ as $\alpha \to \infty$, then, because the Airy function is slowly varying in a neighbourhood of the end-point of integration $w = 0$ and is bounded outside the neighbourhood, it follows from Watson's lemma (Olver 1974, p. 112) that

$$J_2 \sim \mathrm{Ai}\,(\epsilon_2) = A_2. \tag{33}$$

If, however, $|\epsilon_2|$ becomes large as $\alpha \to \infty$, then the Airy function in (32) is rapidly varying near the point of integration $w = 0$, and a different asymptotic analysis is now appropriate: the Airy function is replaced by its asymptotic expansion (30), and the resulting integral is evaluated asymptotically. Rather than perform the calculation in this way, we first, by making a change of variable, express J_2 in the form

$$J_2 = \alpha^2\exp\,(e^{\frac{1}{6}\pi i}\,\alpha^{\frac{2}{3}}\,\epsilon_2)\int_{e^{\frac{1}{6}\pi i}\alpha^{-\frac{4}{3}}\epsilon 2}^{\infty} e^{-\alpha^2 W}\,\mathrm{Ai}\,(e^{-\frac{1}{6}\pi i}\,\alpha^{\frac{4}{3}}\,W)\,dW. \tag{34}$$

Now use (30) to expand the Airy function in the integrand. This can only be done if $\alpha^{\frac{4}{3}}|W|$ is always large in the integrand, and so we require $|\epsilon_2| \gg 1$. In addition we cannot allow the path of integration in the complex W-plane to cross the line $\arg W = -\frac{5}{6}\pi$, since the asymptotic expansion (30) for the Airy functions fails to be valid near there. When (30) is used in (34) we find

$$J_2 \sim \tfrac{1}{2}\pi^{-\frac{1}{2}} e^{\frac{1}{24}\pi i} \alpha^{\frac{5}{3}} \exp(e^{\frac{1}{6}\pi i} \alpha^{\frac{2}{3}} \epsilon_2) \int_{e^{\frac{1}{6}\pi i} \alpha^{-\frac{4}{3}} \epsilon 2}^{\infty} W^{-\frac{1}{4}} \exp(-\alpha^2(W + \tfrac{2}{3}e^{-\frac{1}{4}\pi i} W^{\frac{3}{2}}))\, dW, \tag{35}$$

where $-\frac{5}{6}\pi < \arg W < \frac{7}{6}\pi$. It may be shown that the argument of the exponential in (35) has no stationary points, and so, provided that the contour is a path of descent, Watson's lemma implies that the asymptotic expansion of the integral consists entirely of contributions from the endpoint of integration. By considering the conformal transformation $W \rightarrow W + \frac{2}{3}e^{-\frac{1}{4}\pi i} W^{\frac{3}{2}}$, one can indeed show that a contour can always be chosen to be a path of descent provided that $|\epsilon_2| \gg 1$ and $|\arg \epsilon_2| < \pi$. Thus one finds from (35)

$$J_2 \sim \frac{\frac{1}{2}\pi^{-\frac{1}{2}} \epsilon_2^{-\frac{1}{4}} \exp(-\frac{2}{3}\epsilon_2^{\frac{3}{2}})}{1 + e^{-\frac{1}{6}\pi i} \alpha^{-\frac{2}{3}} \epsilon_2^{\frac{1}{2}}}.$$

Now since $|\epsilon_2|$ is large and $|\arg \epsilon_2| < \pi$, this result may alternatively be written

$$\frac{J_2}{A_2} \sim \left[1 + \left(1 - i\frac{c}{\alpha^2}\right)^{\frac{1}{2}}\right]^{-1}, \tag{36}$$

where
$$-\frac{2\pi}{3} < \arg\left(1 - i\frac{c}{\alpha^2}\right)^{\frac{1}{2}} < \frac{\pi}{3}.$$

In summary, J_2 has the asymptotic expansion (33) when $\epsilon_2 = O(1)$ and (36) when $|\epsilon_2| \gg 1$. Reid (1979) has derived a uniformly valid (and rather more complicated) expansion for an integral such as (34): however, we do not require such a general result. For the integral J_1, one may similarly deduce the results

$$J_1 \sim \mathrm{Ai}(\epsilon_1) = A_1 \tag{37}$$

when
$$\epsilon_1 = O(1),$$

and
$$\frac{J_1}{A} \sim \left[1 + \left(1 - i\frac{mc}{\alpha^2}\right)^{\frac{1}{2}}\right]^{-1}, \tag{38}$$

where
$$-\frac{\pi}{3} < \arg\left(1 - i\frac{mc}{\alpha^2}\right)^{\frac{1}{2}} < \frac{2\pi}{3}$$

when $|\epsilon_1| \gg 1$.

It is clear from the preceding discussion that we must make *a priori* assumptions about the asymptotic behaviour of c as $\alpha \rightarrow \infty$ if we are to make progress with the asymptotic analysis. We consider four specific regimes; namely, that as $\alpha \rightarrow \infty$

(*a*) $c/\alpha^2 = o(1)$,

(*b*) $c/\alpha^2 \sim -i$,

(*c*) $c/\alpha^2 \sim -im^{-1}$,

(*d*) $c/\alpha^2 \sim \lambda$ for some constant $\lambda \neq -i$ or $-im^{-1}$.

We do not claim that these assumptions represent the only conceivable asymptotic behaviour of the eigenvalues c, merely that these are regimes in which one does find eigenvalues, and that moreover each of the numerically computed eigenvalues found by the methods discussed in §4.2 corresponds, as $\alpha \rightarrow \infty$, to an eigenvalue in one of the asymptotic regimes (*a*)–(*d*).

It is found to be more convenient to consider the secular equation (22) in the following form:

$$4(1-m)^2\frac{J_1}{A_1}\frac{J_2}{A_2}+2(1-m)\left(m+\frac{m}{c}+m\frac{A_2'}{A_2}\right)\frac{J_1}{A_1}$$

$$+2(1-m)\left(-1+\frac{m}{c}+\frac{A_1'}{A_1}\right)\frac{J_2}{A_2}+m\left(2+\frac{1-m}{c}\right)\left(\frac{A_1'}{A_1}-\frac{A_2'}{A_2}\right)=0. \quad (39)$$

It may be seen that there are four functions whose asymptotic expansions we require, namely

$$\frac{J_1}{A_1},\ \frac{J_2}{A_2},\ \frac{A_1'}{A_1},\ \frac{A_2'}{A_2}.$$

We have found expansions for the first two of these in (33), (36)–(38). The asymptotic expansions of the Airy function and its derivative (Abramowitz & Stegun 1965, 10.4.59, 10.4.61) show that the latter two have asymptotic expansions

$$\frac{A_1'}{A_1}\sim-\left(1-i\frac{mc}{\alpha^2}\right)^{\frac{1}{2}},\quad -\frac{\pi}{3}<\arg\left(1-i\frac{mc}{\alpha^2}\right)^{\frac{1}{2}}<\frac{2\pi}{3}, \quad (40a)$$

$$\frac{A_2'}{A_2}\sim\left(1-i\frac{c}{\alpha^2}\right)^{\frac{1}{2}},\quad -\frac{2\pi}{3}<\arg\left(1-i\frac{c}{\alpha^2}\right)^{\frac{1}{2}}<\frac{\pi}{3} \quad (40b)$$

as $\alpha\to\infty$.

(*a*) $c/\alpha^2 = o(1)$. We find, from (36), (38), (40*a*, *b*) that in this case

$$\frac{J_1}{A_1},\frac{J_2}{A_2}\sim\tfrac{1}{2},\quad \frac{A_1'}{A_1}\sim-1,\quad \frac{A_2'}{A_2}\sim 1. \quad (41)$$

If one multiplies through the secular equation (39) by c and uses these results, one finds, to within the order of approximation implicit in (41),

$$c=0. \quad (42)$$

To improve on this result one has to find higher-order terms in the four asymptotic expansions of (41). Thus one finds

$$\frac{A_1'}{A_1}=-1+\frac{imc}{2\alpha^2}-\frac{im^2}{4\alpha^2}+o(\alpha^{-2}), \quad (43a)$$

$$\frac{A_2'}{A_2}=1-\frac{ic}{2\alpha^2}-\frac{i}{4\alpha^2}+o(\alpha^{-2}), \quad (43b)$$

when, as we are assuming, $c = o(\alpha^2)$. To find higher-order terms in the asymptotic expansion of J_2, consider (35) again, but now with higher-order terms in the integrand. Then use Watson's lemma again to arrive at the result

$$\frac{J_2}{A_2}=\tfrac{1}{2}+\frac{ic}{8\alpha^2}+\frac{i}{8\alpha^2}+o\left(\frac{1}{\alpha^2}\right). \quad (43c)$$

Similarly one finds

$$\frac{J_1}{A_1}=\tfrac{1}{2}+\frac{imc}{8\alpha^2}-\frac{im^2}{8\alpha^2}+o\left(\frac{1}{\alpha^2}\right). \quad (43d)$$

When (43*a*–*d*) are used, one finds, instead of (42),

$$c(-(1+m)^2+O(\alpha^{-2}))=\tfrac{1}{2}im(1-m)^2(1+m)\alpha^{-2}-\tfrac{3}{4}i\frac{c}{\alpha^2}\frac{1-m^2}{m}+o(\alpha^{-2}).$$

and so

$$c\sim i\frac{m(1-m)^2}{2(1+m)}\alpha^{-2}. \quad (44)$$

This is the same result as we derived in §3 by a different method (13*c*)). It describes the asymptotic behaviour of the eigenvalue of the group 1 mode of table 2.

(*b*) $c/\alpha^2 \sim -i$. We again have to find the appropriate asymptotic expansions for the functions that appear in the secular equation (39). From (38) we find

$$J_1/A_1 \sim [1+(1+m)^{\frac{1}{2}}]^{-1},$$

from (33)

$$J_2/A_2 \sim 1,$$

from (40*a*)

$$A_1'/A_1 \sim -(1+m)^{\frac{1}{2}},$$

while A_2'/A_2 cannot be approximated at all in this regime. Thus the secular equation reduces to

$$\frac{A_2'}{A_2} \sim \frac{(1-m)^{\frac{1}{2}}(m^2-2m+2)-2(1-m)^2}{m(1-2m-(1-m)^{\frac{3}{2}})}.$$

Now as $\alpha \rightarrow \infty$, one will find solutions of this equation close to the values of c for which A_2 is zero; i.e.

$$c \sim -i\alpha^2 + e^{\frac{1}{6}\pi i} a_q \alpha^{-\frac{2}{3}} \quad (q = 1, 2, 3, \ldots), \tag{45}$$

where a_q is the qth zero of the Airy function Ai. It is well known that all the zeros are real and negative. They are tabulated in Abramowitz & Stegun (1965, table 10.13). The results (45) describe the asymptotic behaviour of the eigenvalues of the group 3 modes of table 2.

(*c*) $c/\alpha^2 \sim -im^{-1}$. This case is entirely analogous to that just discussed. Instead of (45), one finds

$$c \sim -i\frac{\alpha^2}{m} + e^{\frac{5}{6}\pi i} a_q \alpha^{\frac{2}{3}} m^{-\frac{1}{3}} \quad (q = 1, 2, 3, \ldots). \tag{46}$$

These results describe the asymptotic behaviour of the eigenvalues in the group 2 modes of table 2.

(*d*) $c/\alpha^2 \sim \lambda\ (\neq -i, -im^{-1})$. In this regime we use the asymptotic results (36), (38), (40*a*, *b*) to approximate the functions that appear in the secular equation. The results (36) and (38) may alternatively be written

$$\frac{J_1}{A_1} \sim -i\left[\frac{1-\left(1-i\frac{mc}{\alpha^2}\right)^{\frac{1}{2}}}{mc}\right], \quad \text{where} \quad -\frac{\pi}{3} < \arg\left(1-i\frac{mc}{\alpha^2}\right)^{\frac{1}{2}} < \frac{2\pi}{3}, \tag{47a}$$

$$\frac{J_2}{A_2} \sim -i\left[\frac{1-\left(1-i\frac{c}{\alpha^2}\right)^{\frac{1}{2}}}{c}\right], \quad \text{where} \quad -\frac{2\pi}{3} < \arg\left(1-i\frac{c}{\alpha^2}\right)^{\frac{1}{2}} < \frac{\pi}{3}. \tag{47b}$$

One thus finds, to the order of approximation implicit in these asymptotic results, that the secular equation to determine c becomes

$$-4\alpha^4(1-m)^2\left[\frac{1-\left(1-i\frac{mc}{\alpha^2}\right)^{\frac{1}{2}}}{mc}\right]\left[\frac{1-\left(1-i\frac{c}{\alpha^2}\right)^{\frac{1}{2}}}{c}\right]$$
$$+4i\alpha^2\frac{1-m}{c}\left[\left(1-i\frac{mc}{\alpha^2}\right)^{\frac{1}{2}}-\left(1-i\frac{c}{\alpha^2}\right)^{\frac{1}{2}}\right]-2m\left[\left(1-i\frac{mc}{\alpha^2}\right)^{\frac{1}{2}}+\left(1-i\frac{c}{\alpha^2}\right)^{\frac{1}{2}}\right]=0. \tag{48}$$

Repeated squaring of (48) yields

$$\gamma^6-8(1+m)\gamma^5+16(2+m+2m^2)\gamma^4-8(1+m)(9m^2-10m+9)\gamma^3$$
$$+16(1+m)^2(6m^2-11m+6)\gamma^2-64(1-m)^2(1+m)^3\gamma+16(1-m)^2(1+m)^4=0, \tag{49}$$

m	Root 1	Root 2
0·1	$-0{\cdot}385-i7{\cdot}595$	$15{\cdot}4 \quad -i29{\cdot}29$
0·4	$-0{\cdot}189-i2{\cdot}266$	$2{\cdot}561-i11{\cdot}354$
0·5	$-0{\cdot}159-i1{\cdot}899$	$1{\cdot}707-i9{\cdot}994$
0·6	$-0{\cdot}129-i1{\cdot}651$	$1{\cdot}138-i8{\cdot}954$
0·9	$-0{\cdot}032-i1{\cdot}241$	$0{\cdot}170-i7{\cdot}201$

TABLE 3. The roots c/α^2 of (48) for different values of m

where $\gamma = imc/\alpha^2$. Equation (49) has six roots, but it is found that for all m only two of these roots satisfy (48). The values of c/α^2 for various m corresponding to these two roots are listed in table 3. The group 4 and 5 modes of table 2 correspond respectively to roots 1 and 2 of table 3.

Except for the eigenvalue corresponding to the root 2 solution in table 3, we found satisfactory agreement as α increases between the asymptotic estimates (44), (45), (46), table 3, and the previously computed numerical results (see table 4).

For the root 2 solution in table 3, we found only qualitative agreement. For example for $m = 0{\cdot}5$ and $\alpha = 0{\cdot}4$ our numerical results yielded $c = 0{\cdot}0616-i1{\cdot}3984$ in the κ_{-1} mode, and the root-finding procedure was very inefficient; the asymptotic estimate root 2 in table 3 implies $c = 0{\cdot}273-i1{\cdot}599$. However, the agreement seems good enough for us to identify the modes as the same.

Consider now the asymptotic behaviour of the eigenfunctions. The results derived earlier in this section show that, when the eigenvalues have the asymptotic behaviour (44), the coefficients a_1 and b_1 of (23) are such that as $\alpha \to \infty$

$$a_1 \sim -iA_1 A_2 \frac{1+m}{2(1-m)}\alpha^2, \quad b_1 \sim iA_2 \frac{1+m}{1-m}\alpha^2.$$

Next, if one considers its asymptotic expansion, one may show that the Airy function that appears in the integrand of (23) is such that when $c/\alpha^2 = o(1)$

$$A_1(z) \sim e^{-z} A_1$$

as $\alpha \to \infty$, provided that $z/\alpha^2 \ll 1$. One may legitimately use this result in (23), provided that $y/\alpha^2 \ll 1$. When this is done, one finds

$$\phi_1 \sim iA_1 A_2 \frac{1+m}{1-m}\, \alpha^2 y\, e^{-y}.$$

Similarly one may show that, provided that $-y/\alpha^2 \ll 1$,

$$\phi_2 \sim -iA_1 A_2 \frac{1+m}{m(1-m)} \alpha^2 y\, e^{y}.$$

These results agree with ($10a, b$), the zeroth-order eigenfunction found in §3. Their derivation implies that ($10a, b$) and higher-order corrections are valid for $|y| \ll \alpha^2$, a result that one expects because of the form of the right-hand sides of the Orr–Sommerfeld equations ($4a, b$).

Finally, we consider the eigenfunctions whose eigenvalues have the asymptotic behaviour (35) or (36). Each of these is essentially the same, and we direct our attention to (35). In such modes, A_2 and J_2 vanish to leading order, while A_1 and J_1 do not. On using the representation (19) one therefore finds that to leading order ϕ_1 is zero and ϕ_2 is not, that is the upper fluid is not perturbed and the lower fluid

α	Asymptotic	Numerical
	κ_0, the unstable mode	
1	0·04167	0·022138
2	0·006562	0·005925
3	0·002226	0·002171
4	0·001033	0·001025
5	0·000570	0·000568
6	0·000351	0·000350
7	0·000232	0·000232
	κ_{-2} mode	
α	Asymptotic	Numerical
1	−2·1691	−2·4558
2	−3·6890	−3·8542
3	−3·4959	−5·5898
4	−8·5187	−8·5757
5	−9·7189	−9·7570
6	−12·0718	−12·0989
7	−14·5596	−14·5820
	κ_{+2} mode	
α	Asymptotic	Numerical
1	−2·9279	−3·5489
2	−5·9679	−6·4127
3	−9·5814	−9·9491
4	−13·6270	−13·9472
5	−18·0276	−18·3240
6	−22·7333	−22·9941
7	−27·7089	−27·9492
	κ_{+1} mode	
α	Asymptotic	Numerical
1	−1·899	−2·265
2	−4·785	−5·292
3	−8·216	−8·613
4	−12·058	−12·388
5	−16·236	−16·523
6	−20·704	−20·959
7	−25·429	−25·659

TABLE 4. Comparison between the asymptotic and numerical results for $\mathrm{Im}\,\kappa_n(\alpha)$ as α increases; in each case $m = 0{\cdot}5$

is. Thus, asymptotically as $\alpha \to \infty$, the upper fluid behaves like a solid and the perturbed flow in the lower fluid is that of one of the modes associated with perturbation of the Couette flow of a semi-infinite fluid in the presence of a solid boundary. The analogous situation obtains for the other class of eigenfunctions, those whose eigenvalues have the asymptotic behaviour (36).

5. The energy equation for the unstable mode

It is now clear from the results of §4 that the mode considered in §3 is the only unstable mode, and further that the instability only occurs for sufficiently short wavelength. Thus in discussing the nature of the instability, it is appropriate to confine our attention to the asymptotic behaviour as $\alpha \to \infty$ of the particular mode studied in §3.

A direct and simple explanation of the instability we have found would be of great value, since the instability mechanism might well be the same as that appearing in other, and perhaps mathematically less tractable, problems. We have been unable to find a mechanism that provides such a simple and direct insight. However, energy considerations do provide some insight into the nature of the instability.

If one forms the scalar product of (u_1, v_1) with the terms of the linearized Navier–Stokes equations for the upper fluid, and integrates over the upper fluid, one obtains

$$\int_{\text{upper fluid}} \left(\left(\frac{\partial}{\partial t} + my\frac{\partial}{\partial x} \right) \frac{u_1^2+v_1^2}{2} + mu_1 v_1 + 2\alpha^2 \frac{r}{m} e_{ij}^{(1)} e_{ij}^{(1)} \right) dx\, dy = -\int_{-\infty}^{\infty} [u_1 T_{12}^{(1)} + v_1 T_{22}^{(1)}]_{y=0}\, dx \quad (50)$$

(cf. Batchelor 1967, §3.4). This is the energy equation for the upper fluid. Here u_1 and v_1 are the dimensionless velocity components of the disturbance, while $T_{12}^{(1)}$ and $T_{22}^{(1)}$ are components of the stress tensor of the perturbed flow in the upper fluid:

$$T_{11}^{(1)} = -rp_1 + 2\alpha^2 \frac{r}{m}\frac{\partial u_1}{\partial x}, \quad (51a)$$

$$T_{12}^{(1)} = \alpha^2 \frac{r}{m}\left(\frac{\partial u_1}{\partial y} + \frac{\partial v_1}{\partial x} \right), \quad (51b)$$

$$T_{22}^{(1)} = -rp_1 + 2\alpha^2 \frac{r}{m}\frac{\partial v_1}{\partial y}, \quad (51c)$$

p_1 denoting dimensionless pressure. The symbol $e_{ij}^{(1)}$ in (50) denotes the rate-of-strain tensor of the perturbed flow (which can be derived formally from the expressions for the stress tensor in (51) by setting $p_1 = 0$, $2\alpha^2 r/m = 1$). The three terms in the integrand on the left-hand side of (50) represent respectively the material rate of change of the kinetic energy of the perturbed flow, the rate at which the Reynolds stress is transferring energy between the basic flow and the perturbed flow, and the rate of viscous dissipation of the perturbed flow. The expression on the right-hand side represents the rate at which the disturbed flow in $y > 0$ is being supplied with energy at its boundary $y = 0$. Likewise one finds

$$\int_{\text{lower fluid}} \left(\left(\frac{\partial}{\partial t} + y\frac{\partial}{\partial x} \right) \frac{u_2^2+v_2^2}{2} + u_2 v_2 + 2\alpha^2 e_{ij}^{(2)} e_{ij}^{(2)} \right) dx\, dy = \int_{-\infty}^{\infty} [u_2 T_{12}^{(2)} + v_2 T_{22}^{(2)}]_{y=0}\, dx \quad (52)$$

where $e_{ij}^{(2)}$ and $T_{ij}^{(2)}$ are respectively the rate-of-strain and stress tensors of the perturbed flow in the lower fluid. The components of the stress tensor are

$$T_{11}^{(2)} = -p_2 + 2\alpha^2 \frac{\partial u_2}{\partial x}, \quad (53a)$$

$$T_{12}^{(2)} = \alpha^2 \left(\frac{\partial u_2}{\partial y} + \frac{\partial v_2}{\partial x} \right), \quad (53b)$$

$$T_{22}^{(2)} = -p_2 + 2\alpha^2 \frac{\partial v_2}{\partial y}. \quad (53c)$$

When (50) and (52) are added together one finds that the right-hand sides combine to yield

$$\int_{-\infty}^{\infty} [(u_2 - u_1)\, T_{12}]_{y=0}\, dx. \quad (54)$$

(Note that tangential and normal stress, and normal velocity, are continuous at the interface.) One thus deduces that the sum of the material rate of change of kinetic energy and the rate of viscous dissipation in the perturbed flow throughout the fluids is balanced by the sum of the rate at which the Reynolds stresses transfer energy from the basic flow to the perturbed flow throughout the fluids and the contribution (54). This latter expression must therefore represent the transfer of energy from the basic flow to the perturbed flow at the interface.

Since the quantities that occur in the energy equation can be expressed as descending power series in $\alpha^2, \alpha^0, \alpha^{-2}, \alpha^{-4}, \ldots$, it follows that one can write down equations describing the energy balance at different levels of approximation.

There are only two terms in the energy equation that results from considering the coefficient of α^2: the viscous dissipation of the zeroth-order perturbed flow and the rate of working of the surface stresses at zeroth order. From the results of §3 one finds that the leading contribution to (54) as $\alpha \to \infty$ is

$$\alpha^2 \int_{-\infty}^{\infty} \mathrm{Re}\,[(b_0' - a_0' + 2a_0)\, e^{ix}]\, \mathrm{Re}\,[(b_0'' + 2b_0' + 2b_0)\, e^{ix}]\, dx,$$

i.e.

$$\lambda_0^2 \alpha^2 (1+m) \int_{-\infty}^{\infty} \cos^2 x\, dx.$$

This last integral of course diverges, but one may infer from it the rate of working per wavelength, namely

$$\pi \lambda_0^2 \alpha^2 (1+m). \tag{55}$$

It may be verified directly that this quantity is also the total rate of viscous dissipation of the zeroth-order flow in both fluids. Clearly, therefore, the energy that drives the instability must arise in the higher-order terms.

The next terms to be considered are those that multiply α^0. One finds that they all vanish when integrated over x.

We then proceed to those terms in the energy equation that multiply α^{-2}. It is found that to the rate of working of the surface stresses per wavelength found in (55) one has to add

$$\pi \lambda_0^2 \alpha^{-2} \frac{11m^6 + 51m^5 - 112m^3 + 51m + 11}{16(1+m)} \tag{56}$$

(which is always positive). In addition, the Reynolds stresses supply energy to the perturbed flow at the rate

$$\pi \lambda_0^2 \alpha^{-2}\, \tfrac{1}{2} m^3 (-(1-m) - \tfrac{1}{2} m^2 y^2 - \tfrac{1}{3} m^2 y^3)\, e^{-2y} \quad (y > 0), \tag{57a}$$

$$\pi \lambda_0^2 \alpha^{-2}\, \tfrac{1}{2} (m(1-m) - \tfrac{1}{2} y^2 + \tfrac{1}{3} y^3)\, e^{2y} \quad (y < 0) \tag{57b}$$

per wavelength. We see from (57*a*) that the energy transfer is stabilizing for all y in the more-viscous fluid (i.e. the Reynolds stress transfers energy out of the perturbed flow). In the less-viscous fluid the energy transfer is destabilizing near the interface but is stabilizing far from the interface. When (57*a*, *b*) are integrated over both fluids we find that the rate of transfer of energy to the perturbed flow is

$$-\pi \lambda_0^2 \alpha^{-2}\, \tfrac{1}{8} (1 - 2m + 2m^2 + 2m^3 - 2m^4 + m^5) \tag{58}$$

per wavelength. This is always negative (the terms in the brackets may be written in the form $((1-m)^2 + m^2 + m^3 + m^3(1-m)^2)$. It is clear from (56) and (58) that the energy transfer at the interface is essential in driving the instability.

The analysis we have just described may also be applied to the case of two fluids of unequal density and non-zero surface tension. The conclusions are qualitatively the same.

6. Conclusion

We have shown that the unbounded Couette flow of two viscous fluids of equal density and zero surface tension is always unstable. The instability occurs at the interface of the two fluids and manifests itself in the high-viscosity regime (that is, the disturbance occurs on a lengthscale of the order of, or smaller than, the lengthscale associated with the diffusion of momentum or vorticity). From this it follows that the same instability occurs at the interface of the fluids in any shear flow. Surface tension is always stabilizing; difference in density may be stabilizing or destabilizing. Energy considerations show that the instability is essentially created in the immediate neighbourhood of the interface and in the less-viscous fluid near the interface.

The typical growth rate associated with the instability is comparatively slow, and seems to be sensitive to comparatively low values of surface tension. It seems likely that this is why it has not been observed experimentally. The most relevant experimental work we have found is that of Charles & Lilleleht (1965) and Kao & Park (1972). Both studies were of cocurrent laminar Poiseuille flow of oil and water in a rectangular channel. The instabilities they reported do not seem to arise at the interface: instead the interface deforms in response to instabilities that arise elsewhere.

Finally, we remark that the more general case of two-fluid plane Couette flow in a channel may be tackled by the same methods. We have given this some consideration: the methods are capable of straightforward extension to this problem, but the analysis is inordinately complicated.

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